On *Principia* I.17

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I Significance

Proposition 17 of *Principia* Book I represents not only the culmination of the previous eleven (or more) propositions, but also the beginning of a transition from those propositions into the very different structure of Book III. The preceding series of propositions exhaustively analyzes different possible orbits and establishes their respective force laws. In Prop. 17, this format is turned on its head; we are given the force law and shown the derivation of the orbit. The proposition is not merely a converse to propositions 11-13, though: the character of our givens indicates a more radical change in preparation for Book III. We are given in Proposition 17 not only a specific velocity for a body at a specific position, but also the “absolute quantity” of the force deflecting it. Prop. 17 is the first use of that term in Book I; it is a significant departure from the typical terminology of Newton's previous approach, strictly general and often limited to proportions. Though the proposition cannot be said to be either practical or empirical (it is not at all clear how we might observe in the heavens the “absolute quantity” of a centripetal force, called for by the enunciation) it begins to permit in principle the extrapolation of orbits from data about the orbiting body. It begins, too, to focus our concern on a concrete relationship between one body — a specific body, whose characteristics we observe — and its related center of forces. Our thoughts are gently guided into the vein of empirical astronomy. Newton himself seems to see the proposition as a bridge between the mathematical edifice constructed in Book I and the empirical analysis of Book III. In the introduction to the latter book, he suggests that readers may begin Book III after completing Book I Section III — the section which concludes with Proposition 17.

For many reasons, then, Proposition 17 is of great importance to the system constructed in the *Principia*. It is well worth our while to struggle through the dense mathematics of Prop. 17 and come to a real understanding of how it works. This shall be our initial project, following which we will demonstrate the application of the proposition to contrived data in order to highlight its character as a transition into “The System of the World”.

II Mathematics

The enunciation of Proposition 17 runs thus:

> Supposing that the centripetal force is inversely proportional to the square of the distance of places from the center and that the absolute quantity of this force is known, it is required to find the line which a body describes when going forth from a given place with a given velocity along a given straight line.[1]

We can boil this enunciation down a bit, permitting ourselves a bit of anachronistic terminology in the interest of clarity. We are to find the “line” (assumed, by Prop. 13 cor. 1, to be a conic section) described by a body acted on by a centripetal force F, given:

1. I. Bernard Cohen et al., *The Principia*, p. 470

- $F \propto \frac{1}{d^2}$;
- The position of the center of forces S;
- The position and velocity vector of our arbitrary body;
- The "absolute quantity" of the force.

The last given, the absolute quantity of the force, requires some explication. At the outset of the proof Newton assumes this given in the form of an arbitrary orbital segment pq generated by the given force $\propto \frac{1}{d^2}$. In order to actually generate such an orbit, we would need an 'absolute' quantity rather than a proportion — that is, $F = k \times \frac{1}{d^2}$ rather than $F \propto \frac{1}{d^2}$; hence given the orbit we are given the absolute force.

It is probably easiest to take this orbit, as Newton does in the variant proof of Proposition 17 in the *De motu corporum in gyrum*, as a circle. The purpose of this imaginary orbit is to give us the principal latus rectum of the real orbit, via the proportion

$$L_1 : L_2 :: SY^2 \times V^2 : sy^2 \times v^2 \tag{1}$$

proved in Prop. 16 cor. 1. Both velocities are assumed given; L_{Circle} is equal to the given radius sp; sy is equal to the distance sp (tangents to circles are perpendicular to the radii through the points of tangency); and SY is given through $\triangle SYP$, in which angles $\angle SPY$ and $\angle SYP$ and side PS are given. Thus L_{Conic} is given and can be computed thus:

$$L_{Conic} = \frac{SY^2 \times V^2 \times L_{Circle}}{sy^2 \times v^2} \tag{2}$$

If we do not make the simplifying assumption that our imaginary orbit is a circle, sy is given for the same reasons as SY, while the latus rectum can be computed as $\frac{qt^2}{qr}$. These latter quantities are also given — qr is the deflection from straight-line motion, while qt is ultimately equal to the (given) velocity. Hence, whatever our imaginary orbit should be, we can by Prop. 16 cor. 1 obtain the principal latus rectum of the real orbit.

Now we must show that enough of the structural parameters of the conic section are given to permit us to define the section. In order to maintain a completely general proof, we will obtain that definition by means of the principal latus rectum and the locations of the foci. We already have the principal latus rectum, L, and one focus, S; we will therefore seek the position of the other focus, H.

Focus S is given by hypothesis. We erect the line MPH such that $\angle RPH$ is supplemental to $\angle RPS$, thus making $\angle ZPH$ (extending RP to Z) equal to $\angle RPS$. Per Apollonius 3.48, the other focus of the figure lies somewhere on line MPH, though it may be on either side of the tangent RPZ.

Newton then asks us to add the conjugate semiaxis BC and drop SK perpendicular to PH. Two of the notable features of this proposition here rear their heads. Firstly, we are going to be working with lines and magnitudes whose structural relationships to our givens are known, but whose actual positions and magnitudes are unknown. Here, we do not know the location of C or the length

Figure 1: Ellipse

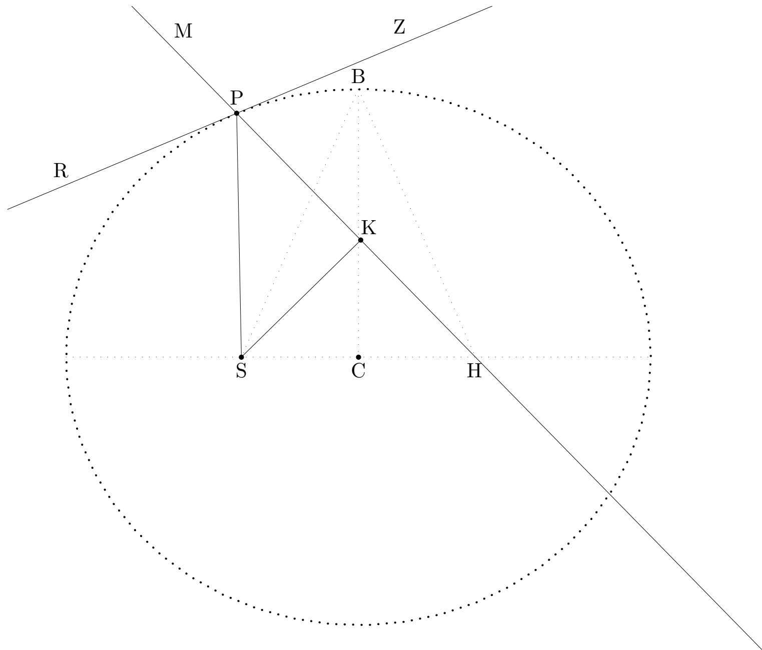


Figure 2: Hyperbola

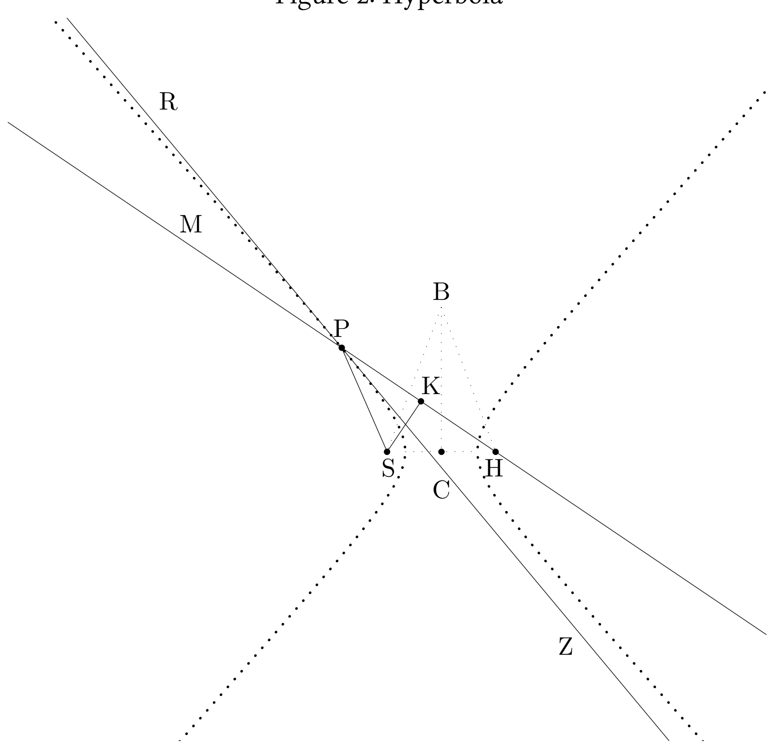
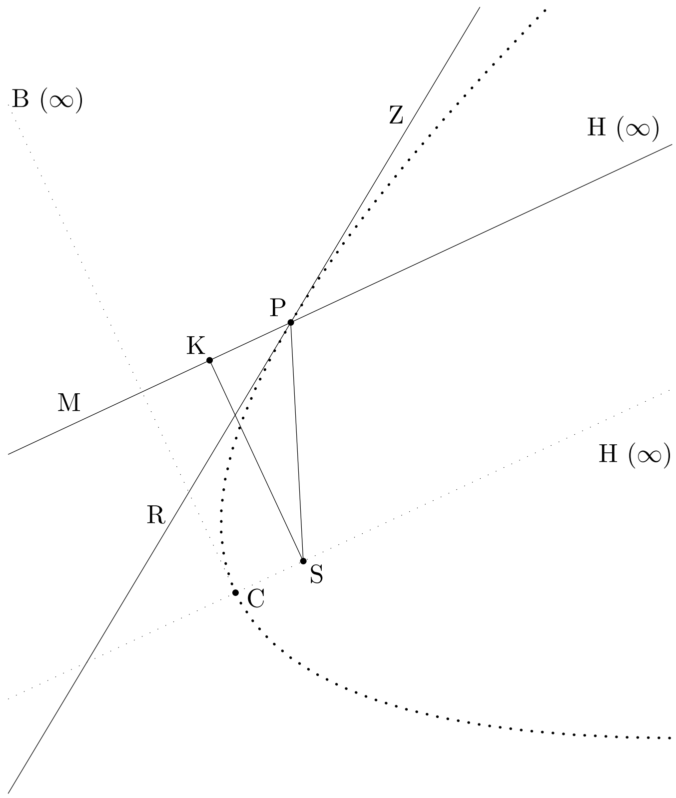


Figure 3: Parabola



of the semi-minor axis, but we *do* know a number of relationships from Apollonius subsisting between the semi-minor axis and other known and unknown features of the conic sections. Hence we assume line BC for algebraic purposes and can use it to determine relationships between our knowns and unknowns, but cannot use it in calculation. The second oddity here presented is the extremely strange construction of the diagram for the parabola (see figure 3). In the parabolic case, C must be taken as the principal vertex, the parabola lacking a center (the principal vertex is however analogous, being located at a fixed distance from the given focus S and midway between that focus and the directrix). Apollonius would give us the length of the conjugate semiaxis from I Def. 11: First diameter : Second diameter :: Second diameter : L. Taking the rectangles of means and extremes, however, this indicates that the second diameter BC is infinite. This infinity of otherwise-finite lines will be a consistent feature of the parabolic proof, but it does allow the algebra to work out with a modicum of mental stretching.

Now, construction complete, we must determine the length PH along the line MPH to find the other focus. Newton begins thus:

$$PS^2 - 2 \times KP \times PH + PH^2 = SH^2 \tag{3}$$

This equation is a bit abstruse; Newton has silently simplified a more basic relationship for us. The equation he presumably began with is

$$PS^2 + PH^2 - 2 \times PS \times PH \times \frac{PK}{PS} = SH^2 \tag{4}$$

which we recognize as equivalent to the Law of Cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$) applied to $\triangle PSH$. Considering the cosine function simply as a ratio of adjacent side to hypotenuse, the relationship is easily provable (in full generality) with Euclid.

For the first of many times, we must prove the same algebraic statement in two different ways to maintain the full generality of the proof. Newton wants to show that $SH^2 = 4BH^2 - 4BC^2$.

Ellipse and Hyperbola $SC = CH = \frac{1}{2}SH$ (foci equidistant from center), making $SH^2 = 4CH^2$. Invoking Euclid I.47, $4CH^2 = 4BH^2 - 4BC^2$.

Parabola $SH = CH = \infty$, making $SH^2 = 4CH^2$. $\triangle BCH$ is right, since BC, the "conjugate axis," is perpendicular to the principal axis. Although $CH = BC = BH = \infty$, we assert by the Pythagorean Theorem that $4CH^2 = 4BH^2 - 4BC^2$.

Thus

$$PS^2 - 2 \times KP \times PH + PH^2 = 4BH^2 - 4BC^2 \tag{5}$$

We will now find a substitution for both $4BH^2$ and $4BC^2$. Firstly, we will show that $4BH^2$ — the distance between the focus H and the endpoint of the semi-minor axis — is equal to $(PS + PH)^2$. There are two cases for this equality:

Ellipse and Hyperbola $SC = CH$; $BC = BC$; therefore $BS = BH$, $BH = \frac{1}{2}(BS + BH)$ and $4BH^2 = (BS + BH)^2$. In either the ellipse or the hyperbola, $BS + BH = PS + PH$, from their locus definitions. In the hyperbola, one of the magnitudes in each pair will be negative.

Parabola In the parabola, the other focus is at an infinite distance along the principal axis. Hence, PS is finite; $BH = PH = \infty$; therefore $4BH^2 = (PS + BH)^2 = (PS + PH)^2 = \infty$.

Similarly we must show that $4BC^2 = L \times (PS + PH)$.

Ellipse and Hyperbola In both cases, $PS + PH =$ Major axis. (In the case of the hyperbola, either PS or PH will be negative, making the expression a difference of the distances). From Apollonius, I Def. 11, Major axis : Minor axis :: Minor axis : L. Therefore (Minor axis)^2 = $L \times$ (Major axis). Since $4BC^2 =$ (Minor axis)^2 (as BC is the semi-minor axis), $4BC^2 = L \times (PS + PH)$.

Parabola $BC = PH = \infty$, while PS and L are finite; hence $4BC^2 = L \times (PS + PH)$.

Substituting these values into equation 5 and expanding the binomial $(PS + PH)^2$, we have

$$PS^2 - 2 \times KP \times PH + PH^2 = (PS^2 + 2 \times PS \times PH + PH^2) - L \times (PS + PH) \tag{6}$$

Canceling common terms and solving for $L \times (PS + PH)$, we obtain

$$L \times (PS + PH) = 2 \times PS \times PH + 2 \times KP \times PH \tag{7}$$

Dividing both sides by $L \times PH$, we obtain

$$\frac{(PS + PH)}{PH} = \frac{2 \times (PS + KP)}{L} \tag{8}$$

or in proportional notation, as Newton concludes,

$$(PS + PH) : PH :: 2 \times (PS + KP) : L \tag{9}$$

PS, KP, and L are given. Hence the ratio $2 \times (PS + KP) : L$ is given in all cases. The value of this ratio gives us an easy way to determine which conic section we are dealing with.

Ellipse If $2 \times (PS + KP) > L$, then $(PS + PH) > PH$, that is, both magnitudes are positive and the expression $PS + PH$ is a sum of positive magnitudes. The figure, by the locus definition, is therefore an ellipse.

Hyperbola If $2 \times (PS + KP) < L$, then $(PS + PH) < PH$, that is, one of our magnitudes is negative and the expression $PS + PH$ is a difference of magnitudes. The figure is therefore a hyperbola.

Parabola If $2 \times (PS + KP) = L$, then $(PS + PH) = PH$. PS is given and presumed not to be zero (the centripetal force having in that case no deflective effect); hence $PH = \infty$. The figure is therefore a parabola, whose second focus lies at an infinite distance.

In the cases of the hyperbola and the ellipse, we can further show from this ratio that PH is given, and hence that we can describe the figure. Taking the above proportion *separando*:

$$PS : PH :: 2 \times (PS + KP) - L : L \tag{10}$$

Hence:

$$PH = \frac{L \times PS}{2 \times (PS + KP) - L} \tag{11}$$

And PH may be found by computation, since all of the quantities on the right side are given.

In the case of the parabola, the axis will be parallel to PK (or PH) through the given focus S. In the conception of the parabola as infinitely extended ellipse, the second focus lies at an infinite distance on a line SH through the given focus. We have shown that the focus H lies on the line PH, at an infinite distance; lines PH and SH are therefore parallel as they meet at infinity. Since the angle of line PH to the given line RPZ is given by construction, we know the angle of the diameters of the parabola and can construct the axis at this angle through the given center of forces S. The parabola is therefore given.

Q. E. F.

III Application

Finally, let us assume some simple, contrived "orbital data" and demonstrate the use of Proposition 17 to extrapolate the imaginary body's orbit. We will for simplicity's sake represent our quantities on a Cartesian coordinate plane and assign our givens thus:

- $F \propto \frac{1}{d^2}$. Absolute quantity of force known; let this force be represented by a circle of radius 10 units, on which the body would move at a speed of 8 units per second.
- Center of force S located at the origin (0, 0).
- Body located at (0, 12).
- Body's velocity 5 units per second along a line at 60° to the horizontal through (0, 12)

We can calculate the latus rectum of our conic via equation 2. Rewriting this equation in line with our discussion of the circular absolute-force orbit and substituting our knowns, we obtain

$$L_{Conic} = \frac{SY^2 \times (5u/s)^2 \times 10u}{(10u)^2 \times (8u/s)^2} \tag{12}$$

Table 1: Known magnitudes

Name	**Value**
sy	$10u$
ps	$10u$
v	$8u/s$
PS	$12u$
V	$5u/s$
SY	(see eq. 13)

SY is given by our knowledge of triangle $\triangle SYP$.

$$SY = \sin\angle SPY \times PS = \sin 30^\circ \times 12u = 6u \tag{13}$$

We can then compute our latus rectum:

$$L_{Conic} = \frac{(6u)^2 \times (5u/s)^2 \times 10u}{(10u)^2 \times (8u/s)^2} = 1.40625u \tag{14}$$

Referring back to equation 9, we see that we require only the magnitude PK to determine with which conic section we are dealing. PK is given via the known triangle $\triangle PSK$, in which we have the hypotenuse PS, $\angle SPH$ (by construction equal to $180^\circ - 2 \times \angle RPS$), and the right angle PKS. Thus

$$PK = \cos(180^\circ - 2 \times \angle RPS) \times PS = \cos 120^\circ \times 12u = 6u \tag{15}$$

We then have all the terms necessary to evaluate eq. 9. The ratio on the right side of the proportion becomes

$$\frac{2 \times (PS + KP)}{L} = \frac{2 \times (12u + 6u)}{1.40625} = \frac{36u}{1.40625u} \tag{16}$$

Since $2 \times (PS + KP) > L$, $(PS + PH) > PH$, making our conic section an ellipse. We therefore know that we can evaluate PH meaningfully (it is a finite magnitude), and apply equation 11:

$$PH = \frac{1.40625u \times 12u}{2 \times (12u + 6u) - 1.40625u} \approx 0.4878u \tag{17}$$

The figure is now completely determined: it will be an ellipse whose major axis $PS + PH \approx 12.4878u$ and whose principal latus rectum will be $1.40625u$; and we can describe this figure. Since line PH is inclined 30° from our given velocity vector, assumed at 60° from horizontal, PH is at 30° from horizontal. We can easily calculate this line's slope:

$$m = \frac{\sin 30°}{\cos 30°} = \frac{1}{\sqrt{3}} \tag{18}$$

And, since we know that the line's y-intercept is 12 (see Table 1), we can write an equation for this line.

$$y = \frac{1}{\sqrt{3}}x + 12 \tag{19}$$

We know that H is located $0.4878u$ along this line in the positive x direction from P. Given the slope of the line PH, we can calculate the position of H on the coordinate plane. The paradigm 30°-60°-90° triangle has sides in the proportion $1 : \frac{1}{2} : \frac{\sqrt{3}}{2}$. Since the distance along the line, equal to the hypotenuse, is $0.4878u$, we can obtain the x and y increments from the point (0, 12) known to lie on the line by multiplying the other sides in the unit triangle by $0.4878u$. By adding these increments to (0, 12), we find the coordinates of the point H as (0.4225, 12.2439).

We then have the coordinates of the foci — (0, 0) and (0.4225, 12.2439) — the length of the major axis, and the principal latus rectum. We can therefore plot the ellipse on the coordinate plane (see figure 4).

Which is what was required to be done. *Q. E. F.*

Figure 4: The given ellipse